Exact Differential Equations

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Differential Equations
Exact Differential Equation

If \( f(x, y) \) is a function of two variables with continuous first partial derivatives in a region \( R \) of the xy-plane, then its differential is

\[
d f(x, y) = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]

A first order differential equation

\[
M(x, y) \, dx + N(x, y) \, dy = 0
\]

is called exact differential equation if and only if there exists a function \( f(x, y) \) as above with

\[
\frac{\partial f}{\partial x} = M(x, y), \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y)
\]

In that case we can solve the DE by observing that

\[
d f(x, y) = 0 \Rightarrow f(x, y) = c
\]
Example

Example. Is the differential equation

\[ 3x^2y^2 \, dx + 2x^3 \, y \, dy = 0 \]

an exact DE?
Example

**Example.** The differential equation

\[ 3x^2y^2\,dx + 2x^3y\,dy = 0 \]

is an exact DE, because for \( f(x, y) = x^3y^2 \) we have

\[ \frac{\partial}{\partial x} (x^3y^2) = 3x^2y^2 \]

and

\[ \frac{\partial}{\partial y} (x^3y^2) = 2x^3y \]
Example

Example. Is the differential equation

\[ x^2 y^3 \, dx + x^3 y^2 \, dy = 0 \]

an exact DE?
Example

Example. The differential equation

\[ x^2 y^3 \, dx + x^3 y^2 \, dy = 0 \]

is an exact DE, because for \( f(x, y) = \frac{1}{3} x^3 y^3 \) we have

\[
\frac{\partial}{\partial x} \left( \frac{1}{3} x^3 y^3 \right) = x^2 y^3
\]

and

\[
\frac{\partial}{\partial y} \left( \frac{1}{3} x^3 y^3 \right) = x^3 y^2
\]

**Question:** How can we tell that a DE is exact?
Criterion for an Exact Differential

**Criterion:** Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives. Then a necessary and sufficient condition so that $M(x, y)dx + N(x, y)dy = 0$ is an exact DE is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

**Example.** Prove that the differential equations

$$3x^2y^2dx + 2x^3ydy = 0, \quad x^2y^3dx + x^3y^2dy = 0$$

are exact DE.

**Question:** How can we find the functions $f(x, y)$ corresponding to exact DE?
Example

**Example.** Solve the following DE

\[ 2xy \, dx + (x^2 - 1) \, dy = 0 \]
Solution for Exact DE $M(x, y)dx + N(x, y)dy = 0$

1. First we check the validity of the Criterion

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$ 

2. If it holds then there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M(x, y), \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y)$$

3. We integrate the first with respect to $x$ while holding $y$ constant:

$$f(x, y) = \int M(x, y)dx + g(y)$$

4. We then differentiate with respect to $y$ and substitute to the second equation.

5. We solve for $g'(y)$ and integrate with respect to $y$ to find it.
Example

**Example.** Determine if the following DE is exact. If it is solve it

\[ 2xydx + (x^2 - 1)dy = 0 \]
Example

**Example.** Determine if the following DE is exact. If it is solve it

\[(2x + y)dx - (x + 6y)dy = 0\]
Example

**Example.** Determine if the following DE is exact. If it is solve it

\[(5x + 4y)dx + (4x - 8y^3)dy = 0\]
Example

**Example.** Determine if the following DE is exact. If it is solve it

\[(2y - \frac{1}{x} + \cos3x) \frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin3x = 0\]
Example

**Example.** Determine if the following DE is exact. If it is solve it

\[(\sin y - y \sin x)dx + (\cos x + x\cos y - y)dy = 0\]
Example

**Example.** Solve the following IVP

\[(e^x + y)dx + (2 + x + ye^y)dy = 0, \quad y(0) = 1\]
Exercise

Exercise. Solve the following DE

\[(e^{2y} - y\cos(xy))dx + (2xe^{2y} - x\cos(xy) + 2y)dy = 0\]
Exercise

**Exercise.** Solve the following IVP

\[
\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1 - x^2)}, \quad y(0) = 2
\]
Exercises

**Exercises.** Solve Exercises 5-16 and 21-26 from Page 69 of the Textbook